

1 Wiadomości wstępne

- Narysować wykresy funkcji elementarnych: $\sin x$, $\cos x$, $\operatorname{tg} x$, $\operatorname{ctg} x$, a^x ($0 < a \neq 1$), $\log_a x$ ($0 < a \neq 1$), $\arcsin x$, $\arccos x$, $\operatorname{arctg} x$, $\operatorname{arctg} x$. Podać ich dziedziny i przeciwdziedziny.
- Rozłożyć na ułamki proste wyrażenie (wynik sprawdzić)

$$\begin{array}{lll} \text{a)} \frac{8x+7}{x^2+3x-4}, & \text{b)} \frac{x+7}{x^2+5x+6}, & \text{c)} \frac{x+29}{x^2-2x-35}, \\ \text{d)} \frac{5x^2-3x+12}{(x-1)(x^2+x+5)}, & \text{e)} \frac{2x^2+5x+14}{(x+2)(x^2+2x+4)}, & \text{f)} \frac{3x^2-2x+7}{x^3+3x+4}. \end{array}$$

- Rozwiązać równania

$$\begin{array}{llll} \text{a)} \sin x = \sin \frac{\pi}{3}, & \text{b)} \cos x = \cos \frac{4\pi}{3}, & \text{c)} \cos x = \cos(-\frac{\pi}{4}), & \text{d)} \sin x = \sin \frac{7\pi}{6}, \\ \text{e)} \cos x = \frac{1}{2}, & \text{f)} \sin x = \frac{\sqrt{2}}{2}, & \text{g)} \sin x = -\frac{\sqrt{3}}{2}, & \text{h)} \cos x = -\frac{\sqrt{3}}{2}, \\ \text{i)} \sin x = 0, & \text{j)} \sin x = 1, & \text{k)} \sin x = -1, & \text{l)} \cos x = 0, \\ \text{m)} \cos x = 1, & \text{n)} \cos x = -1. & & \end{array}$$

Odp.:

$$\begin{array}{llll} \text{a)} x = \frac{\pi}{3} + 2k\pi \vee x = \frac{2\pi}{3} + 2k\pi, & \text{b)} x = \frac{4\pi}{3} + 2k\pi \vee x = -\frac{4\pi}{3} + 2k\pi, & \text{c)} x = \frac{\pi}{4} + 2k\pi \vee x = -\frac{\pi}{4} + 2k\pi, & \\ \text{d)} x = \frac{7\pi}{6} + 2k\pi \vee x = -\frac{\pi}{6} + 2k\pi, & \text{e)} x = \frac{\pi}{3} + 2\pi k \vee x = -\frac{\pi}{3} + 2\pi k, & \text{f)} x = \frac{1}{4}\pi + 2\pi k \vee x = \frac{3}{4}\pi + 2\pi k, & \\ \text{g)} x = -\frac{1}{3}\pi + 2\pi k \vee x = \frac{4}{3}\pi + 2\pi k, & \text{h)} x = -\frac{5}{6}\pi + 2\pi k \vee x = \frac{5}{6}\pi + 2\pi k, & \text{i)} x = \pi k, & \\ \text{j)} x = \frac{1}{2}\pi + 2\pi k, & \text{k)} x = -\frac{1}{2}\pi + 2\pi k, & \text{l)} x = \frac{1}{2}\pi + \pi k, & \\ \text{m)} x = 2\pi k, & \text{n)} x = \pi + 2\pi k, k \in \mathbb{Z}. & & \end{array}$$

- Rozwiązać układy równań

$$\begin{array}{llll} \text{a)} \begin{cases} \cos x = \frac{1}{2} \\ \sin x = \frac{\sqrt{3}}{2} \end{cases}, & \text{b)} \begin{cases} \cos x = -\frac{1}{2} \\ \sin x = \frac{\sqrt{3}}{2} \end{cases}, & \text{c)} \begin{cases} \cos x = -\frac{1}{2} \\ \sin x = -\frac{\sqrt{3}}{2} \end{cases}, & \text{d)} \begin{cases} \cos x = \frac{1}{2} \\ \sin x = -\frac{\sqrt{3}}{2} \end{cases}, \\ \text{e)} \begin{cases} \cos x = -\frac{\sqrt{2}}{2} \\ \sin x = -\frac{\sqrt{2}}{2} \end{cases}, & \text{f)} \begin{cases} \cos x = -\frac{\sqrt{3}}{2} \\ \sin x = -\frac{1}{2} \end{cases}, & \text{g)} \begin{cases} \cos x = -\frac{\sqrt{3}}{2} \\ \sin x = \frac{1}{2} \end{cases}, & \text{h)} \begin{cases} \cos x = -\frac{\sqrt{2}}{2} \\ \sin x = \frac{\sqrt{2}}{2} \end{cases}, \\ \text{i)} \begin{cases} \cos x = 0 \\ \sin x = 1 \end{cases}, & \text{j)} \begin{cases} \cos x = -1 \\ \sin x = 0 \end{cases}, & \text{k)} \begin{cases} \cos x = 0 \\ \sin x = -1 \end{cases}, & \text{l)} \begin{cases} \cos x = 1 \\ \sin x = 0 \end{cases}. \end{array}$$

Odp.:

$$\begin{array}{llll} \text{a)} x = \frac{\pi}{3} + 2k\pi, & \text{b)} x = \frac{2\pi}{3} + 2k\pi, & \text{c)} x = \frac{4\pi}{3} + 2k\pi, & \text{d)} x = -\frac{\pi}{3} + 2k\pi, \\ \text{e)} x = \frac{3\pi}{4} + 2\pi k, & \text{f)} x = \frac{7}{6}\pi + 2\pi k, & \text{g)} \frac{5}{6}\pi + 2\pi k, & \text{h)} x = \frac{3\pi}{4} + 2\pi k, \\ \text{i)} x = \frac{1}{2}\pi + 2\pi k, & \text{j)} x = \pi + 2\pi k, & \text{k)} x = -\frac{1}{2}\pi + 2\pi k, & \text{l)} x = 2\pi k, k \in \mathbb{Z}. \end{array}$$

- Rozwiązać równania i nierówności

$$\begin{array}{llllll} \text{a)} \log_3 x = \log_3 5, & \text{b)} \log_{\frac{1}{2}} 7 = \log_{\frac{1}{2}} x, & \text{c)} \log x > \log 10, & \text{d)} \log x \leq 1, & \text{e)} \ln x \geq 2, & \text{f)} 4 \geq \log_{\frac{1}{2}} x, \\ \text{g)} \log_{\frac{2}{3}} x \geq -1, & \text{h)} \ln(x-5) = 0, & \text{i)} \log(x+3) > -1, & \text{j)} \log_2(x^2-5) = 2, & \text{k)} \log_{\frac{1}{3}}(x^2+2) > -3. \end{array}$$

Odp.:

$$\begin{array}{llllll} \text{a)} x = 5, & \text{b)} x = 7, & \text{c)} x > 10, & \text{d)} x \in [0, 10], & \text{e)} x \in [e^2, \infty), & \text{f)} x \in [\frac{1}{16}, \infty), & \text{g)} x \in (0, \frac{3}{2}], \\ \text{h)} x = 6, & \text{i)} x \in (-\frac{29}{10}, \infty), & \text{j)} x \in \{-3, 3\}, & \text{k)} x \in (-5, 5). \end{array}$$

2 Granice ciągu

Określić symbol granicy i obliczyć:

1) $\lim_{n \rightarrow \infty} \frac{1}{n + \frac{1}{2}}, \text{Odp. : } 0,$

4) $\lim_{n \rightarrow \infty} \frac{-5n^2 + 7n - 2}{4n - 7}, \text{Odp. : } -\infty,$

7) $\lim_{n \rightarrow \infty} \frac{2n^2 + 6n - 1}{3n - 7}, \text{Odp. : } \infty,$

10) $\lim_{n \rightarrow \infty} (\sqrt{n+2} - \sqrt{n}), \text{Odp. : } 0,$

13) $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n), \text{Odp. : } \frac{1}{2},$

16) $\lim_{n \rightarrow \infty} (\sqrt{3n^2 + 2n - 5} - n\sqrt{3}),$
 $\text{Odp. : } \frac{1}{3}\sqrt{3},$

19) $\lim_{n \rightarrow \infty} \sqrt{n(n - \sqrt{n^2 - 1})}, \text{Odp. : } \frac{\sqrt{2}}{2},$

22) $\lim_{n \rightarrow \infty} \left(\frac{n+5}{n}\right)^n, \text{Odp. : } e^5,$

25) $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n-2}\right)^{2n-1}, \text{Odp. : } e^6,$

28) $\lim_{n \rightarrow \infty} \left(\frac{2n+3}{3n+2}\right)^n, \text{Odp. : } 0,$

31) $\lim_{n \rightarrow \infty} \frac{4^{n-1} - 5}{2^{2n} - 7}, \text{Odp. : } \frac{1}{4},$

34) $\lim_{n \rightarrow \infty} \frac{2^{n+1} - 3^{n+2}}{3^{n+2}}, \text{Odp. : } -1,$

37) $\lim_{n \rightarrow \infty} \frac{9^{\log_3 n}}{4^{\log_2 n}}, \text{Odp. : } 1,$

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2) $\lim_{n \rightarrow \infty} \frac{3n+5}{4n-7}, \text{Odp. : } \frac{3}{4},$

5) $\lim_{n \rightarrow \infty} \frac{7n-2}{4n^2+5n-7}, \text{Odp. : } 0,$

8) $\lim_{n \rightarrow \infty} \frac{5n^2+2n-3}{4n^2-9}, \text{Odp. : } \frac{5}{4},$

11) $\lim_{n \rightarrow \infty} (n - \sqrt{n^2 + 5n}), \text{Odp. : } -\frac{5}{2},$

14) $\lim_{n \rightarrow \infty} \frac{\sqrt{n^2 - 1}}{\sqrt[3]{n^3 + 1}}, \text{Odp. : } 1,$

17) $\lim_{n \rightarrow \infty} (\sqrt[3]{n^3 + 4n^2} - n), \text{Odp. : } \frac{4}{3},$

20) $\lim_{n \rightarrow \infty} (\sqrt{n + \sqrt{n}} - \sqrt{n - \sqrt{n}}),$
 $\text{Odp. : } 1,$

23) $\lim_{n \rightarrow \infty} \left(1 - \frac{4}{n}\right)^{-n+3}, \text{Odp. : } e^4,$

26) $\lim_{n \rightarrow \infty} \left(\frac{n}{n-6}\right)^{2n}, \text{Odp. : } e^{12},$

29) $\lim_{n \rightarrow \infty} \left(\frac{n^2 + 3n - 5}{n^2 + 2n + 2}\right)^{n^2 - 5n + 2},$
 $\text{Odp. : } \infty,$

32) $\lim_{n \rightarrow \infty} \frac{5 \cdot 3^{2n} - 1}{4 \cdot 9^n + 7}, \text{Odp. : } \frac{5}{4},$

35) $\lim_{n \rightarrow \infty} \sqrt[n]{9^n + 8^n + 7^n}, \text{Odp. : } 9,$

38) $\lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}}{1 + \frac{1}{3} + \frac{1}{9} + \dots + \frac{1}{3^n}}, \text{Odp. : } \frac{4}{3},$

3) $\lim_{n \rightarrow \infty} \frac{-5n^2 + 7n - 2}{3n^2 + 4n - 7}, \text{Odp. : } -\frac{5}{3},$

6) $\lim_{n \rightarrow \infty} \frac{2n^3 - n^2 + 5n - 3}{7 - 3n^2}, \text{Odp. : } -\infty,$

9) $\lim_{n \rightarrow \infty} \frac{6n^3 - 5n^2 + 2n - 3}{3n^3 - 4n^2 - 9n + 2}, \text{Odp. : } 2,$

12) $\lim_{n \rightarrow \infty} (3n - \sqrt{9n^2 + 6n - 15}), \text{Odp. : } -1,$

15) $\lim_{n \rightarrow \infty} \frac{\sqrt{1 + 2n^2} - \sqrt{1 + 4n^2}}{n}, \text{Odp. : } \sqrt{2} - 2,$

18) $\lim_{n \rightarrow \infty} (n\sqrt{2} - \sqrt{2n^3 + 5n^2 - 7}), \text{Odp. : } -\frac{5\sqrt{2}}{6},$

21) $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n, \text{Odp. : } e^2,$

24) $\lim_{n \rightarrow \infty} \left(\frac{n^2 + 6}{n^2}\right)^{n^2}, \text{Odp. : } e^6,$

27) $\lim_{n \rightarrow \infty} \left(\frac{2n+1}{2n-1}\right)^n, \text{Odp. : } e,$

30) $\lim_{n \rightarrow \infty} \left(\frac{n^2 + 3n - 5}{n^2 + 2n + 2}\right)^{-5n+2}, \text{Odp. : } e^{-5},$

33) $\lim_{n \rightarrow \infty} \frac{3 \cdot 2^{2n+2} - 10}{5 \cdot 4^{n-1} + 3}, \text{Odp. : } \frac{48}{5},$

36) $\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{2}{3}\right)^n + \left(\frac{3}{4}\right)^n}, \text{Odp. : } e^{\ln 3 - \ln 4},$

3 Granice funkcji

Znaleźć granice funkcji:

$$1) \lim_{x \rightarrow \infty} \frac{1+x-3x^2}{1+x+3x^2}, \text{Odp. : } -1$$

$$4) \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x}, \text{Odp. : } \frac{1}{2}$$

$$7) \lim_{x \rightarrow 1} \frac{x^3+x-2}{x^3-x^2-x+1}, \text{Odp. : undefined}$$

$$10) \lim_{x \rightarrow 1} \left(\frac{1}{1-x} - \frac{3}{1-x^3} \right), \text{Odp. : } -1$$

$$13) \lim_{x \rightarrow -2} \frac{3x^2+5x-2}{4x^2+9x+2}, \text{Odp. : } 1$$

$$16) \lim_{x \rightarrow 5} \frac{\sqrt{x-1}-2}{x-5}, \text{Odp. : } \frac{1}{4}$$

$$19) \lim_{x \rightarrow \infty} x(\sqrt{x^2+1}-x), \text{Odp. : } \frac{1}{2}$$

$$22) \lim_{x \rightarrow 0} \frac{1-\cos x}{x^2}, \text{Odp. : } \frac{1}{2}$$

$$25) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos 2x}, \text{Odp. : } \frac{1}{2}\sqrt{2}$$

$$28) \lim_{x \rightarrow 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\frac{tgx}{x+1}}, \text{Odp. : } 1$$

$$31) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x, \text{Odp. : } 1$$

$$34) \lim_{x \rightarrow \infty} \left(\frac{2x+1}{x-2} \right)^{2x-1} : \infty$$

$$2) \lim_{x \rightarrow \infty} \left(\frac{x^3}{x^2+1} - x \right), \text{Odp. : } 0$$

$$5) \lim_{x \rightarrow 5} \frac{\sqrt{x-1}-2}{x-5}, \text{Odp. : } \frac{1}{4}$$

$$8) \lim_{x \rightarrow -2} \frac{x^3+3x^2+2x}{x^2-x-6}, \text{Odp. : } -\frac{2}{5}$$

$$11) \lim_{x \rightarrow 4} \frac{x^2-2x-8}{x^2-9x+20}, \text{Odp. : } -6$$

$$14) \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2}-1}{x}, \text{Odp. : } 0$$

$$17) \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}, \text{Odp. : } \frac{2}{3}$$

$$20) \lim_{x \rightarrow 0} \frac{\sin 3x}{x}, \text{Odp. : } 3$$

$$23) \lim_{x \rightarrow 0} \frac{tgx - \sin x}{x^3}, \text{Odp. : } \frac{1}{2}$$

$$26) \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) tgx, \text{Odp. : } 1$$

$$29) \lim_{x \rightarrow \infty} \left(\frac{x+1}{x-2} \right)^{2x-1}, \text{Odp. : } e^6$$

$$32) \lim_{x \rightarrow \infty} \left(\frac{3x-4}{3x+2} \right)^{\frac{x+1}{3}}, \text{Odp. : } e^{-\frac{2}{3}}$$

$$35) \lim_{x \rightarrow 0} \frac{\ln(a+x) - \ln a}{x} : \frac{1}{a}$$

$$3) \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}, \text{Odp. : } 1$$

$$6) \lim_{x \rightarrow 1} \frac{x^2-2x+1}{x^3-x}, \text{Odp. : } 0$$

$$9) \lim_{x \rightarrow \frac{1}{2}} \frac{8x^3-1}{6x^2-5x+1}, \text{Odp. : } 6$$

$$12) \lim_{x \rightarrow 1} \frac{x^3+125}{2x^2-50}, \text{Odp. : } -\frac{21}{8}$$

$$15) \lim_{x \rightarrow 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}, \text{Odp. : } 3$$

$$18) \lim_{x \rightarrow 1} \frac{\sqrt{x^2+1} - \sqrt{x+1}}{1 - \sqrt{x+1}}, \text{Odp. : } 0$$

$$21) \lim_{x \rightarrow 0} \frac{tgx}{\sin 5x}, \text{Odp. : } \frac{1}{5}$$

$$24) \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{tgx} \right), \text{Odp. : } 0$$

$$27) \lim_{x \rightarrow \frac{\pi}{6}} \frac{\sin(x - \frac{\pi}{6})}{\frac{\sqrt{3}}{2} - \cos x}, \text{Odp. : } 2$$

$$30) \lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x, \text{Odp. : } e^{-1}$$

$$33) \lim_{x \rightarrow \infty} \left(\frac{x+1}{2x-2} \right)^x, \text{Odp. : } 0$$

$$36) \lim_{x \rightarrow 0} \frac{\ln(1+kx)}{x} : k$$

4 Reguła de L'Hospitala

Obliczyć granice funkcji stosując regułę de L'Hospitala:

1) $\lim_{x \rightarrow 0} \frac{\ln \cos x}{x}, \text{Odp. : } 0$

4) $\lim_{x \rightarrow 0} \frac{x - \sin x}{x - \operatorname{tg} x}, \text{Odp. : } -\frac{1}{2}$

7) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln \sin x}, \text{Odp. : } 1$

10) $\lim_{x \rightarrow 0} \left(\operatorname{ctg} x - \frac{1}{x} \right), \text{Odp. : } 0$

13) $\lim_{x \rightarrow 1} \left(\frac{2}{x^2 - 1} - \frac{1}{x - 1} \right), \text{Odp. : } -\frac{1}{2}$

16) $\lim_{x \rightarrow 0} x^2 e^{\frac{1}{x^2}}, \text{Odp. : } \infty$

19) $\lim_{x \rightarrow \frac{\pi}{2}} (\operatorname{tg} x)^{2x - \pi}, \text{Odp. : } 1$

22) $\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}}, \text{Odp. : } e^2$

2) $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x}, \text{Odp. : } 1$

5) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}, \text{Odp. : } 2$

8) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x}, \text{Odp. : } 2$

11) $\lim_{x \rightarrow 1} \left(\frac{x}{x - 1} - \frac{1}{\ln x} \right), \text{Odp. : } \frac{1}{2}$

14) $\lim_{x \rightarrow 0} x \sin \frac{a}{x}, \text{Odp. : } 0$

17) $\lim_{x \rightarrow 1^-} (1 - x) \ln(1 - x), \text{Odp. : } 0$

20) $\lim_{x \rightarrow 0} x^{\sin x}, \text{Odp. : } 1$

23) $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} \right)^{\sin x}, \text{Odp. : } 1$

3) $\lim_{x \rightarrow 0} \frac{x - \operatorname{arctg} x}{x^3}, \text{Odp. : } \frac{1}{3}$

6) $\lim_{x \rightarrow 0} \frac{\ln \sin 2x}{\ln \sin x}, \text{Odp. : } 1$

9) $\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{x}{\ln x} \right), \text{Odp. : } -1$

12) $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right), \text{Odp. : } 0$

15) $\lim_{x \rightarrow \infty} x(e^{\frac{1}{x}} - 1), \text{Odp. : } \infty$

18) $\lim_{x \rightarrow 1} (1 - x) \operatorname{tg} \frac{\pi x}{2}, \text{Odp. : } \frac{2}{\pi}$

21) $\lim_{x \rightarrow 0} x^{\frac{1}{\ln(e^x - 1)}}, \text{Odp. : } e$

24) $\lim_{x \rightarrow a} \left(2 - \frac{x}{a} \right)^{\operatorname{tg} \frac{\pi x}{2a}}, \text{Odp. : } e^{\frac{2}{\pi}}$